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Holograms of cosmological singularities

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Abstract . Raychaudhuri's classic work led to the result that singularities are generic in General Relativity Among them, space-like or null singularities are puzzling they imply that "time" can have a beginning or end Well known examples are singularities inside black holes and initial or final singularities in expanding or contracting universes In recent times, String Theory is providing new perspectives of such singularities which may lead to an understanding of these in the standard framework of time evolution in quantum mechanics In this article we describe some of these approaches

Keywords ADS/CFT duality, Matrix Theory

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1. Introduction

the 1960's it became clear that classical general relativity predicts that under quite reasonable assumptions about initial conditions and equations of state, space-time develops singularities This realization stemmed from the pioneering work of Raychaudhuri in the mid-1950's and culminated in the celebrated singularity theorems of Hawking and Penrose Singularities signify the breakdown of general relativity It has been suspected for a long time that near singularities the very notions of space and time break down and needs to be replaced by more fundamental structure, and there have been many speculations about such a structure In this article, we will discuss some recent progress in String Theory which provides a concrete framework for a structure for many situations

Some kinds of singularities are easier to understand than others *Time-like* singularities are regions of space which exist for all times – and very often they do not appear mysterious once one understands that the singular nature of the space-time is caused by some object sitting there A proper microscopic description of this object would lead to a resolution of such singularities. *Space-like* or *null* singularities are much more difficult to understand These singularities are not located at some point – one cannot look around and *see* that they are there Rather they just *happen*. A well known example is the singularity inside a neutral black hole – all observers who have crossed the horizon will encounter this singularity in finite proper time They cannot "look ahead" and see that there is a singularity and skirt around them Another example is the big bang singularity of an expanding universe This, too, just *happened* at some time in the past.

In string theory, the basic degrees of freedom are one dimensional extended (rather than pointlike) objects in the regime where the string coupling constant is small The very fact that strings have an extension leads to a resolution of some kinds of singularities – a classic example is an *orbifold* singularity. Because of a finite extension, strings can happily propagate on orbifold space-times, while the time evolution of particles is necessarily singular. This in fact can – and does – happen at the *classical* level There are other timelike singularities which can be cured in string theory at the quantum level – this too is reasonable since after all string theory is a consistent quantum mechanical description of gravity

In the following we will describe some recent attempts to understand space-like singularities One reason why it is difficult to understand a space-like singularity is that this signifies a "beginning" or "end" of time. If we believe that the laws of quantum mechanics can be applied to gravity, this is a troublesome concept. Even though the notion of time in a quantum theory of gravity is rather subtle, no one has been able to make sense of a situation where time just stops. Note that this has nothing to do with the fact that such boundaries of time are often associated with the fact that space-time curvatures diverge at these points. Consider for example normal flat space-time and we simply cut out one half of it by putting in a boundary at some time $t = 0$. This is a trivial example of a geodesically incomplete space-time. The quantum dynamics of some field in this space-time is problematic – one has to impose some final state conditions, and there is no reason why the standard hamiltonian cannot evolve the system beyond $t = 0$. The problem becomes more pronounced if in addition the space-time has curvature singularities at initial or final times.

The key to this understanding is the fact that in string theory dynamical space-time is not fundamental, but an *emergent* concept. Over the past decade we have been able to understand the underlying structure from which space-time emerges in *certain circumstances*. Surprisingly this fundamental structure is a gauge theory without dynamical gravity. And even more suprisingly it turns out that this gauge theory lives in a *lower* number of dimensions. Thus a theory without gravity (like a gauge theory) can encode all the details of a theory with dynamical gravity which however lives in higher number of dimensions - a phenomenon which is now called *holography*.

In this article I will discuss this approach to cosmological singularities which has been reviewed in [1,2] and some more recent work in [25–27]. Instead of trying to compile an exhaustive list of references, I have cited a few review articles which should be regarded as a guide to the original literature in the field. This is by no means the only approach to spacelike and null singularities in string theory. Early attempts to address the question in perturbative string theory is reviewed in [3]. Attempts to study perturbative strings in exact time dependent backgrounds and the problems encountered therein are reviewed in [4] and [2]. More recently, there has been some progress in understanding how a phase of "tachyon condensation" can replace spacelike singularities in worldsheet formulations of string theory. This is reviewed in [5].

2. Faces of holography

The key idea evolved from the classic work of 't Hooft in the 1970's. By mid 1970's it was clear that the correct theory for strong interactions is QCD – a non-abelian gauge theory with gauge group $SU(3)$ coupled to quarks in the fundamental representation. However the quarks are confined into mesons and baryons. It was soon realized that this happens because in QCD electric fields cannot spread out – rather they form flux tubes between quarks and antiquarks, forming mesons; or a flux tube closing onto itself forming a glueball, or three flux tubes emerging from three quarks of different colors joining at a vertex, forming a baryon. This was a satisfying picture of confinement. It qualitatively explained why experimentally hadrons appeared to behave like strings – which gave rise to String Theory in the first place. This begs the question what is the coupling constant of strings, g_s in terms of QCD quantities? This should be a dimensionless parameter and QCD does not have any free dimensionless parameter. In a landmark paper in 1974, 't Hooft argued that the dimensionless parameter can be discovered if generalizes the gauge group in QCD from $SU(3)$ to $SU(N)$ and expand the theory in a power series expansion in $1/N$ with the combination $g_{YM}^2 N$ regarded to be $O(1)$. Each term in this expansion is a sum of an infinite number of feynman diagrams which may be thought of tiling a two dimensional surface. If this surface has h handles and B boundaries the overall power of N is simply N^{χ} where χ is the Euler characteristic $\chi = 2 - 2h - B$. This is precisely how a string perturbation theory ought to look like. A typical amplitude in string theory may be written over two dimensional surface which represents the worldsheet of strings with each surface weighted by a factor of g_s^{χ}. Thus the string coupling constant is precisely $1/N$. In the limit $N \to \infty$ with $g_{YM}^2 N$ fixed these strings are weakly coupled. A useful review of large-N may be found in [6].

Following the work of 't Hooft it was in fact realized that not just gauge theories, but any theory whose fields are in the adjoint representation of a large group like $SU(N)$ should give rise to a string theory in the large-N limit. However the precise details of this string theory turned out to be quite elusive.

Around the same time of 't Hooft's work, Yoneya, Scherk and Schwarz proposed that certain supersymmetric versions of string theory contain gravity and reduce to General Relativity at low energies. The celebrated work of Green and Schwarz in 1984 indicated that such string theories in fact form quantum mechanically consistent ultraviolet completions of the supersymmetric generalization of General Relativity, supergravity†. This indicated that

†Standard textbooks on String Theory include [7] and [8]

one can find descriptions of these string theories in terms of large-N gauge theories, the latter may provide a non-perturbative decription of gravity itself

Developments which took place throughout the late 1980's and 1990's led to concrete realizations of this idea – and led to a surprise. The underlying gauge theory lives in *lower* number of dimensions This is now refered to as *holography* and ties up beautifully with an idea which emerged from black hole thermodynamics

The first concrete formulation of holography was achieved in the theory of closed strings in 1 + 1 dimensions [9–11] Here the holographic theory is the gauged quantum mechanics of a single hermitian matrix – which is a gauge theory in 0 + 1 dimensions. This is the first time it was realized that space-time is an emergent concept At the same time it became clear that space-time itself is an approximate concept which breaks down in suitable circumstances The closed string theory contains gravity (though there is no dynamical graviton) and gravitational interactions are encoded in the gauge theory in a subtle and interesting way

The discovery of duality symmetry (for a review see [12]) and D-branes (for an introduction see *e g.* [13]) led to another realization of holography The low energy dynamics of a stack of N p-dimensional D-branes in string theory is a $U(N)$ gauge theory living on the $p + 1$ dimensional world-volume. However these objects produce gravitational backgrounds This indicates that gravitational phenomena in the *entire* space-time should be describable in this gauge theory In fact the appropriate limit of the gauge theory is precisely 't Hooft's large-N limit Perhaps the most striking result of this development has been the statistical explanation of black hole thermodynamics and Hawking radiation. A certain class of black holes appear as a stacks of a large number of D-branes wrapped in internal directions and appear as states in the gauge theory In certain cases, their degeneracies can be counted reliably This leads to a statistical entropy which is in precise agreement with the results of Bekenstein and Hawking in semiclassical gravity Furthermore, Hawking radiation of such black holes may be understood as usual quantum mechanical decay of excietd states in this gauge theory and the decay rate is in precise agreement with the semiclassical luminosity For reviews of black holes in string theory see [14,15]

This connection between gauge theory and gravity is in fact a manifestation of a basic property of string theory which has been known since its inception . open-closed duality In string theory processes involving open strings can be also viewed as processes involving closed strings Now the low energy limit of open string theory is a gauge theory, while the low energy limit of closed strings contain gravity – so there should be connection here The dynamics of D-branes are described by open strings which live on the brane – the gauge theory in the previous paragraph is in fact the low energy limit of this theory of open strings It has been recently realized that the gauge theory – gravity connection uncovered in 1+1 dimensions is also a result of open-closed duality The matrix of gauged matrix quantum mechanics in fact represents the degrees of freedom of a bunch of D-particles in this theory In the region near horizons of certain black holes, the theory of open strings can be truncated to its low energy limit – which leads to a precise connection between this gauge theory and gravity. This connection is known as the AdS/CFT correspondence since these near-horizon geometries are asymptotically anti-de-Sitter space-times and the holographic gauge theory which may be thought of living on the boundary is some deformation of a conformally invariant field theory A useful review of this dual correspondence is [18] Pretty much like the two dimensional example, an extra space dimension is generated in the gauge theory, and at the same time the theory secretly contains dynamical gravity In the AdS/CFT correspondence, the additional dimension arises out of the ***renormalization group scale*** of the quantum field theory The RG equations of the field theory are essentially Raychaudhuri equation in the bulk (for connections of Raychaudhuri equation with AdS holography see *e g* [16,17]

The open-closed duality applied to D-branes implies a rather different relation between gauge theories and closed string theories which are called "Matrix Theories" This follows from the fact that even though critical string theories live in ten space-time dimensions, these are actually 11-dimensional theories in disguise This 11-dimensional theory is called M-theory. String theories appear as Kaluza-Klein reductions of M-theory on a circle, the momentum along the circle appearing as usual as a charge in the 10 dimensional theory – this charge is in fact the charge associated with D0 branes Now suppose we consider M-theory in an infinite boosted reference frame along a compact direction In this frame the only states which survive are those which carry a quantized positive momentum J along the direction of boost. From the point of view of string theory, these are states which carry a large D0 brane charge J Now, boosting along a compact spatial circle leads to a compact ***null*** circle It turns out that one can go to a regime where the effective theory is the $U(J)$ gauged matrix quantum mechanics which describe the low energy limit of the theory of D0 branes Furthermore if additional dimensions are compact this matrix quantum mechanics effectively becomes low dimensional supersymmetric Yang-Mills theories on compact spaces This Yang-Mills theory then describes string theory and hence garvity in ten dimensional space-time Like the previous two examples, space dimensions are "manufactured" in the gauge theory The reason why this gauge-string connection is different from the previous two examples is that here we do not need a 't Hooft limit In fact a strong version of the Matrix theory conjecture works for any finite J A review of Matrix Theory is [19]

In these examples, a gauge theory leads to gravitational theories in higher number of *space* dimensions The *time* was built in Nevertheless, as we will see below, the notion of time perceived by the open strings, *i e* the gauge theory can be quite different from the notion of time perceived by closed strings which emerge out of the fundamenatal theory There are in fact versions of Matrix Theory in which even time is manufactured – the holographic theory is simply a random matrix theory For a review see [20]

Except in the two dimensional string theory example, the gauge-gravity correspondence re-main conjectures rather than proven relationships The common theme in all these realizations of holography is that the quantum dynamics of the system is unambiguously defined in the holographic gauge theory The higher dimensional space-time and gravitational dynamics in it is always an *effective* description in a certain regime of parameters By the same token, usual *perturbative* closed string theory would appear only in a certain corner of the parameter space A "proof" of the conjecture would involve a demonstration that in this corner of parameter space, the gauge theory indeed reproduces closed string perturbation theory This is the sense in which there is such a proof for the two dimensional model It is confusing to ask what a "proof would mean at the non-perturbative level – since we do not know of an independent way to define closed strings non-perturbatively In fact, it seems reasonable to take the open string theory as a fundamental definition of the theory itself This theory has no dynamical space-time, no notion of geodesic incompleteness and so on – but does have a notion of time evolution of states In a certain approximation, the theory can be re-interpreted as a theory of perturbative closed strings In general, there is no such interpretation and therefore no notion of dynamical space-time.

This is the key point in our discussion of singularities. These holographic descriptions lead to a proposal for the structure which replaces space-time near singularities ***If there are situations where the fundamental dynamics defined in terms of the gauge theory makes sense in regions where the space-time appears singular one has the possibility that these singularities are simply problems with interpretation.***

3. Null singularities and matrix strings and membranes

Our first set of examples concern ten dimensional string theory in suitable backgrounds which appear to be singular but which have holographic descriptions in terms of "Matrix Theory"

3 1 Matrix string theory .

Let us briefly review the holographic description of Type IIA strings in terms of a 1 + 1 dimensional supersymmetric Yang-Mills theory (for a useful review see [21] This connection arises from duality transformations on the standard ten dimensional flat background with string frame metric

$$ds^2 = 2dx^+dx^- + d\mathbf{x}\, d\mathbf{x}, \tag{1}$$

with the null coordinate x^- compactified on a circle of radius R. The string coupling is g_s and the string length is l_s. Consider the sector of the theory with a momentum

$$p_- = J/R \tag{2}$$

along x^-

Now, ten dimensional string theory has a rich set of symmetries called duality symmetries which generally relate one kind of string theory with another kind. In this case the above background may be shown to be dual to a Type IIB string theory with a string coupling $\tilde{g}_s$ and string length $\tilde{l}_s$ given by

$$\tilde{g}_s = \frac{R}{g_s l_s},\ \tilde{l}_s^2 = \frac{g_s l_s^3}{R}, \tag{3}$$

living on a compact space-like circle of radius $\tilde{R}$ given by

$$\tilde{R} = \frac{l_s^2}{R}, \tag{4}$$

and carrying J units of D1-brane charge The Matrix theory conjecture in this case claims that a non-perturbative formulation of the theory is given by a $U(J)$ Yang-Mills theory with a dimensional coupling constant

$$g_{YM} = \frac{R}{g_s l_s^2} \tag{5}$$

The bosonic part of the action is given by

$$S = \int d\tau d\sigma Tr\left\{\frac{1}{9} g_s^2 F_{\tau\sigma}^2 + \frac{1}{2}(D_\tau X^i)^2 - \frac{1}{2}(D_\sigma X^i)^2 + \frac{1}{4g_s^2}[X^i, X^j]^2\right\} \qquad (6)$$

where X^i, $i = 1 \ldots 8$ are adjoint scalars and $F_{\tau\sigma}$ is the $U(J)$ gauge field strength. These fields live on a circle parametrized by σ and $0 < \sigma < \tilde{R} = l_s^2/R$. However the space-time of this Yang-Mills theory is not dynamical.

In the regime of weak string coupling $g_s << 1$ the potential term in (6) suppresses X^i which do not commute. Therefore in this limit the theory reduces to a theory of $8J$ scalar fields which may be chosen to be the diagonal components of the matrices X^i, which we denote by X^i_n, $n = 1 \ldots J$. Similarly, the gauge field strength is locally zero. However as we go around the σ circle, the gauge symmetry allows non-trivial boundary conditions which are characterized by conjugacy classes of the gauge group. For example we can have

$$X_1^i(\sigma + 2\pi\tilde{R}) = X_2^i(\sigma)$$
$$X_2^i(\sigma + 2\pi\tilde{R}) = X_3^i(\sigma)$$
$$\ldots$$
$$X_J^i(\sigma + 2\pi\tilde{R}) = X_1^i(\sigma) \qquad (7)$$

With these boundary conditions, the action (6) becomes the action of 8 massless scalars living on a circle of radius $J\tilde{R}$. This is precisely the worldsheet description of a single Type IIA string in the light cone gauge. In a similar way one could have boundary conditions

$$X_1^i(\sigma + 2\pi\tilde{R}) = X_2^i(\sigma)$$
$$X_2^i(\sigma + 2\pi\tilde{R}) = X_3^i(\sigma)$$
$$\ldots$$
$$X_K^i(\sigma + 2\pi\tilde{R}) = X_1^i(\sigma)$$
$$X_{K+1}^i(\sigma + 2\pi\tilde{R}) = X_{K+2}^i(\sigma)$$
$$X_{K+2}^i(\sigma + 2\pi\tilde{R}) = X_{K+3}^i(\sigma)$$
$$\ldots$$
$$X_J^i(\sigma + 2\pi\tilde{R}) = X_{K+1}^i(\sigma) \qquad (8)$$

and one would have the worldsheet action for *two* strings. It is clear that for a given J one would have various sectors of the theory which describes J strings. Furthermore the commutator interaction is capable of describing in a precise fashion joining and splitting of these strings.

Therefore in the $g_s << 1$ regime, this two dimensional gauge theory describes a second quantized theory of closed strings in *ten space-time dimensions*, and therefore gravitational interactions in ten dimensions as well. It is important to realize that this happens only in this regime, which by virtue of (5) is the strongly coupled regime of the gauge theory. The fields X^i metamorphize into transverse space-time coordinates. For generic g_s all the non-abelian excitations are important and there is no such ten dimensional interpretation and therefore no clear interpretation in terms of a usual dynamical space-time.

3.2 *A matrix big bang*.

It turns out that there is a remarkably simple modification of flat space which provides a useful model of a cosmological singularity [22]. This is a background which still has a flat string frame metric, but has in addition a dilaton which is linear in the null coordinate x^+

$$\Phi = -Qx^+ \qquad (9)$$

For $Q > 0$ the point $x^+ = -\infty$ is in fact a *null big bang singularity*. This is because the Einstein frame metric (*i.e.* in terms of which the low energy effective theory is the Einstein-Hilbert action) is geodesically incomplete, with geodesics reaching $x^+ = -\infty$ in a finite proper time. The effective string coupling $g_{eff} = e^{\Phi}$ is infinite at this point. In a similar way for $Q < 0$ we have a big crunch.

The logic which leads to a 1 + 1 dimensional Yang-Mills theory in the $p_- = J/R$ sector now leads to a theory whose bosonic action is

$$S = \int d\tau d\sigma \mathrm{Tr}\left\{\frac{1}{2} g_s^2 e^{-2Q\tau} F_{\tau\sigma}^2 + \frac{1}{2}(D_\tau X^i)^2 - \frac{1}{2}(D_\sigma X^i)^2 + \frac{1}{4g_s^2} e^{2Q\tau}[X^i, X^j]^2\right\} \qquad (10)$$

Thus the effect of the non-trivial dilaton is to make the coupling constant of the Yang-Mills theory time dependent. It is useful to rewrite this action (10) as follows

$$S = \int d\tau d\sigma \sqrt{h} \mathrm{Tr}\left\{\frac{1}{4} g_s^2 h^{ab} h^{cd} F_{ac} F_{bd} + \frac{1}{2} h^{ab}(D_a X^i)(D_b X^i) + \frac{1}{4g_s^2}[X^i, X^j]^2\right\} \qquad (11)$$

where h_{ab} denotes a two dimensional metric

$$ds_2^2 = h_{ab} d\xi^a d\xi^b = e^{2Q\tau}[-d\tau^2 + d\sigma^2] \qquad (12)$$

Thus the theory may be viewed as one with *constant* couplings, but in a nontrivial space-time (12) – which is the Milne universe, or the future light cone of the origin We would expect that in an appropriate limit the action (11) represents a string theory living in this Milne space-time with 8 other transverse directions. It is clear from (12) that the space-time defined by (τ, σ) with $-\infty < \tau < \infty$, $0 < \sigma < 2\pi\tilde{R}$ has a conical singularity at $\tau = -\infty$ and this singularity may be reached in a finite proper time One way to see the geodesic incompleteness of the space-time is to go to Minkowskian coordinates for the metric

$$T = e^{Q\tau}\cosh\sigma \quad X = e^{Q\tau}\sinh\sigma$$
$$ds_2^2 = -dT^2 + dX^2 \tag{13}$$

The situation is quite similar to the two dimensional backgrounds of the previous section There is an *open string time* τ which runs over the full range However the closed string time T seems to begin at $T = 0$ at a null big bang

From the action (10) we see that at $\tau \to \infty$, *i.e* $T \to \infty$ the Yang-Mills coupling is strong Threfore we expect that this regime would describe perturbative IIB strings in the manner discussed above However, as $\tau \to -\infty$, *i e* at the "big bang" the Yang-Mills coupling is *weak* This means that there is nothing which suppresses X^i which are non-commuting and all the $8J^2$ degrees of freedom are equally important This means that there is no intepretation in terms of second quantized strings and no interpretation of the fields as coordinates in eight transverse space dimensons The Yang-Mills theory, however, continues to make perfect sense In fact the coupling of the theory goes to *zero* as we approach this "singularity" and nothing is obviously wrong with a bunch of free fields !

What has happened is similar to the two dimensional example There is a certain regime of the parameters of the holographic theory where a space-time interpretation is valid If we forcibly extraploate this interpretation to early or late times, it appears that from the point of view of the closed string theory there is a singularity However it is precisely in this region that it is illegal to ascribe a space-time interpretation the holographic gauge theory is what it is and makes perfect sense.

3.3 *Zooming onto the big bang* .

At the place where the closed string theory appears to perceive a big bang singularity the non-ableian nature of the theory becomes important. To get some idea of the nature of non-ableian excitations it is useful to consider strings moving on a time dependent gravitational wave rather than flat space [23] Fortunately Matrix Theory may be forumlated in a class of such gravitational waves whose string frame metric and dilaton are given by

$$ds^2 = 2dx^+dx^- - \left[\left(\frac{\mu}{3}\right)^2 x^2 + \left(\frac{\mu}{6}\right)^2 y^2\right]\times$$
$$\left(dx^+\right)^2 + dx\cdot dx\,dy\; dy,\ \Phi = -Qx^+ \tag{14}$$

where $x = (x^1 \cdots x^3)$, $y = (y^1 \cdots y^5)$ There is, in addition, a background 5-form field strength also proportional to μ The resulting matrix theory is a deformation of the theory described in the previous subsection additional terms involving μ

The introduction of a gravitational wave as above introduces a new length scale $1/\mu$ into the Yang-Mills theory It turns out that the dimensionless ratio μ/G_{YM} acts as a semiclassical parameter and when $\mu >> G_{YM}$ *classical* solutions of the Yang-Mills theory are relevant Among such classical solutions are highly non-abelian configurations called *fuzzy spheres*

$$X^i = S(\tau)J^i \quad i = 1,\ 2,\ 3 \tag{15}$$

where $S(\tau)$ is a real function of τ and J^i are generators of a N-dimensional respresentation of $SU(2)$. These are called spheres since the Casimir condition implies that

$$\sum_{i=1}^{3}(X^i)^2 = \frac{N^2-1}{4}S^2(\tau)I_{N\times N} \tag{16}$$

which would represent a sphere if the X^i were real numbers, rather than matrices They are called fuzzy becuase X^i are not real numbers Clearly in the presence of a large number of fuzzy spheres, the matrices are far from being diagonal and the usual interpretation of the theory as a string theory in conventional ten dimensional space-time is invalid These fuzzy spheres are really discretized versions of D-branes

The time dependence of the radius of the fuzzy sphere, $S(\tau)$ is determined by the dynamics of the YM theory An examination of the action shows that at early times (near the big bang) such fuzzy spheres rather than strings proliferate. As time evolves, the size of these fuzzy spheres diminish and at late times they become zero size objects leaving only strings

A Type IIB version of this model [24] exhibits another interesting aspect of the region near the singularity It is well known that quantum field theory in time dependent backgrounds exhibit particle production. This results from the fact that typically the vacuum state at late times is not vacuum at early times and *vice versa* In these models, same phenomenon leads to an interesting insight into question of initial conditions. We saw that in this type model one generically expects perturbative strings and conventional space-time at late times and nonabelian configurations at early times The latter include D-branes which can be easily excited The question we can now ask the following If we require that the state at late times contain only perturbative strings and nothing else, can we get with *any* conceivable initial state?

One might expect that the answer should be positive that all at late times these D-branes and other non-abelian excitations are suppressed – we might expect that they all go away at sufficiently late times regardless of the initial state Surprisingly the answer is negative Because particle production (or depletion) effects, the state which does not contain such D-branes at late times turns out to be a *squeezed state* of these D-branes at early times in a thermal distribution of the number of D-branes temperature $1/Q$. This means that if we require that at late times usual space-time emerges with no such remnant branes filling the vacuum – the initial state near the big bang must be close to this special squeezed state It will be interesting to see whether this has some implication for the general question of initial conditions in cosmology

Null singularities in AdS/CFT correspondence

There is another class of "singular" backgrounds which can be analyzed using the standard AdS/CFT correspondence [25–27] Once again we will see that while the gravity description becomes singular, the open string description in terms of the dual gauge theory remains perfectly well defined

The usual AdS/CFT correspondence

The simplest setting the AdS/CFT correspondence states that Type IIB string theory in a $AdS_5 \times S^5$ background is equivalent, or dual, to a $\mathcal{N} = 4$ supersymmetric gauge theory in 3 + 1 dimensions In this case, the latter gauge theory lives on the boundary of AdS_5. There are a large number of generalizations and extensions of this holographic correspondence In this article, however, this simplest setting is adequate. Unlike the prvious examples involving Matrix Theory or earlier 2d examples, we do not yet know how to start from the gauge theory and make a duality transformation to obtain the string theory Rather, we know the correspondence to a large extent through a dictionary of a large class of operators in the two descriptions Specifically, gauge invariant operators in the gauge theory create states which are identified with states of the string theory

The gauge theory has two parameters – the gauge coupling g_{YM} and the rank of the gauge group N. These are related to the two parameters characterizing the string theory in this background – the AdS scale in string units R/l_s and the string coupling constant g_s by

$$g_s = g_{YM}^2 \qquad \frac{R}{l_s} = (4\pi g_{YM}^2 N)^{1/4} \tag{17}$$

These relations immediately show that the string theory is weakly coupled if $g_{YM} << 1$ On the other hand, the AdS space-time has a non-vanishing scale in string units only if $g_{YM}^2 N$ is finite Therefore we need to perform the limit $g_{YM} \to 0$, $N \to \infty$ with $g_{YM}^2 N$ finite – *i e.* the 't Hooft limit For a fixed AdS scale, the $1/N$ expansion becomes the string perturbation expansion Finally when $g_{YM}^2 N >> 1$ - *i.e* when the gauge theory is strongly coupled, we have $R >> l_s$ and the dual string theory can be well approximated by its low energy limit – supergravity

$\mathcal{N} = 4$ Yang-Mills theory in 3 + 1 dimensions is conformally invariant (the beta function vanishes) and has in addition a internal R symmetry whose bosonic part is $SO(6)$ The conformal symmetries are realized in the dual description as *isometries* of AdS_5, while the R symmetries are realized as *isometries* of the S^5 This then is an example in which dynamical ten dimensional space-time containing gravity has emerged out of a 3 + 1 dimensional theory containing no gravity ***In fact, since the ten dimensional space-time makes sense only when $R \gg l_s$ such an interpretation is valid only when the gauge theory is strongly coupled.*** Generically the gauge theory makes sense, but we cannot describe the physics in terms of a ten dimensional relativistic space-time

In the following we will use Poincare coordinates in $AdS_5 \times S^5$. The einstein frame metric is given by

$$ds^2 = \frac{r^2}{R^2}\left[2dx^+ dx^- + dx^2\right] + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 \tag{18}$$

where the null coordinates $x^\pm$ and the two space-like coordinates x form a $R^{(1,3)}$ The radial coordinate r has a range $0 \le r \le \infty$ and $d\Omega_5^2$ denotes the standard metric on a S^5. The boundary of this space-time is the $R^{(1,3)}$ at $r = \infty$. In addition to the nontrivial metric the background also has a background five form field strength given by

$$F_5 = R^4(\omega_5 + *_{10}\omega_5) \tag{19}$$

All other fields vanish Since the dilaton is trivial, (18) is also the string frame metric

In terms of these coordinates the gauge theory is defined on a $R^{(1,3)}$ with a flat Minkowskian metric

4.2. Time dependent and null deformations .

It turns out that the above solution can be extended to an infinite class of solutions given by

$$ds^2 = \frac{r^2}{R^4}\left[g_{\mu\nu}(x)dx^\mu dx^\nu \right] + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2$$

$$F_5 = R^4(\omega_5 + *_{10}\omega_5) \tag{20}$$

where $x^\mu = (x^\pm, x)$ and $g_{\mu\nu}$ is a function of these four x^μ This is a solution of the supergravity equations of motion provided in addition there is a background dilaton $\Phi(x)$ which is purely a function of x^μ and the following equations are satisfied

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\,\partial_\nu\Phi$$

$$\nabla^2\Phi = 0 \tag{21}$$

where $R_{\mu\nu}$ is the Ricci tensor computed from the metric $g_{\mu\nu}$, and ∇ is the covariant derivative defined using this same metric

It is easy to find solutions to (21) These include solutions with space-like singularities of the Kasner form For reasons which will become clear soon, we will consider a special set of solutions where the metric $g_{\mu\nu}$ is conformally flat with a conformal factor which is a function of one of the null coordinates x^+ and the dilaton is also a function of x^+

$$g_{\mu\nu} = e^{f(x^+)}\eta_{\mu\nu} \quad \Phi(x) = \Phi(x^+) \tag{22}$$

The conditions (21) then becomes

$$\frac{1}{2}(f')^2 - f'' = \frac{1}{2}(\partial_+\Phi)^2 \tag{23}$$

Thus we can obtain an infinite number of solutions of this type, by specifying any function $f(x^+)$ and solving for $\Phi(x^+)$

4.3. The gauge theory duals :

There is a natural candidate for the gauge theory dual of a background of the form (20) Note that if we regard this solution as a deformation of the standard $AdS_5 \times S^5$, it is a *non-normalizable* deformation Therefore, according to the standard rules of AdS/CFT correspondence this should be dual to the $\mathcal{N} = 4$ gauge theory deformed by sources In this case, it is quite plausible that the dual theory is in fact the original gauge theory living on a 3 + 1 dimensional space-time with the metric $g_{\mu\nu}$ and a position dependi coupling $e^{\Phi(x)}$. This is fairly obvious when $g_{\mu\nu}$ is a sm deformation of $\eta_{\mu\nu}$. For finite deformations the argum which justifies this claim is similar to the one which led the AdS/CFT correspondence in the first place, discussed in [25,27]

The null solutions conformal to flat space particularly useful in this context This is because confor invariance of the $\mathcal{N} = 4$ gauge theory ensures that conformal factor decouples at the classical level Howe on curved space-time this theory typicaly has a confor anomaly It turns out that when the conformal factor i function of x^+ alone, the conformal anomaly vanishes well This means that the only non-trivial difference betw this deformed gauge theory and the original gauge the is the x^+ dependent coupling constant

4.4. Gauge theory dual of a big crunch / big bang

A particularly illuminating example where the gravity-ga theory duality can be used to gain insight is a confor null background of the form (22) with

$$e^{f(x^+)} = \tanh^2 x^+$$

The solution to the eq (23) is

$$' = g_s \tanh \frac{\hat{}}{}|$$

This solution has the following features

(i) This space-time has a null singularity at x^+ where tidal forces diverge and geodesics end finite proper time

(ii) e^Φ and hence the coupling of the string th *vanishes* at this singularity

(iii) At $x^+ \to \pm\infty$ the metric becomes $AdS_5 \times S^5$ \ a constant dilaton $e^\Phi = g_s$

The crucial feature of this solution that the coupling bounded everywhere and vanishes at the singularity we evolve in the light cone time x^+ one has a $AdS_5 \times$ which shrinks to a big crunch, followed by a big bang again a $AdS_5 \times S^5$ in the future In classical superg it is meaningless to go through the singularity

Let us now examine the behavior of the gauge the dual It is convenient to do this in light front quantiza where x^+ is regarded as time Since the supergra background is $AdS_5 \times S^5$ at $x^+ \to -\infty$, the gauge th starts out in the standard conformally invariant vacuum the past As we have argued before, the nontrivi comes from the x^+ dependent dilaton. In fact, for exan the gauge part of the action reads

$$S = \int d^4x\, \mathrm{Tr}\, e^{\Phi(x^+)} F_{\mu\nu}F^{\mu\nu} \tag{26}$$

the prefactor diverges at the singularity one might that typical correlators in the gauge theory also diverge Indeed there are gauge invariant operators in this theory whose correlators do diverge.

Nevertheless we will now argue that the gauge theory is actually completely non-singular Note that the basic fields A_μ in (26) do not have a normalized kinetic energy term. To examine the behavior of the theory, we first need to normalize the kinetic term This can be easily done in cone gauge

$$A_- = 0 \tag{27}$$

redefining fields as

$$\tilde{A}_i = e^{\Phi/2} A_i, \quad \tilde{A}_+ = e^{\Phi/2} A_+, \tag{28}$$

where the index i in the first equation above takes values in the directions transverse to x^+, x^-. This gives

$$-\frac{1}{4}\int d^4x \Big[\mathrm{Tr}\left(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu\right)^2$$
$$-2ie^{\Phi/2}\mathrm{Tr}\left\{\left(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu\right)\left[\tilde{A}^\mu, \tilde{A}^\nu\right]\right\}$$
$$-e^{\Phi}\mathrm{Tr}\left(\left[\tilde{A}_\mu, \tilde{A}_\nu\right]\right)^2 - \partial x - \left\{\left(\partial_+\Phi\right)\tilde{A}_i\tilde{A}^i\right\}\Big] \tag{29}$$

In the last term the index i takes two values transverse to x^+, x^- We see that *since the dilaton depends only* the last term is a total derivative in x^-, it will not affect the equations of motion and can be dropped The dilaton has now disappeared from the quadratic terms, but reappear in the interaction terms. However it is easy to check that such interaction terms always contain *positive* powers of e^{Φ} and therefore *vanish* at $x^+ = 0$

These new variables $\tilde{A}_\mu$ *are a complete set of gauge invariant variables (these are gauge invariant because we have fixed the gauge completly).* These can be used to define a standard Fock space Since their interactions $x^+ = 0$, all correlations of these fields are well defined and smooth at all times. In other words, any gauge invariant state constructed by superposing Fock space states with smooth coefficients will evolve in a smooth ...lar fashion through the singularity into the second region Therefore the region $x^+ = 0$ which appeared as a null singularity in the gravity description is a regular region from the gauge theory point of view Finally there is no particle production, which follows from the fact that the background depends only on x^+.

The gauge theory therefore provides a nonsingular time evolution of the system. At $x^+ \to \pm\infty$ this theory has a useful interpretation in terms of supergravity and therefore conventional space-time. Near $x^+ = 0$, however, the gauge theory is *weakly coupled* Therefore according to the AdS/CFT wisdom, supergravity is not a good approximation of the dual closed string It remains to be seen whether perturbative string theory makes sense in this region

It is interesting to note that operators constructed out of the fields A do not generally have duals which create supergravity modes at all times For example the operator which is dual to a dilaton in the bulk is given by $\mathrm{Tr}e^{-\Phi}F^2$, and correlators of this quantity can be singular or even ambiguous at $x^+ = 0$ This is not a problem since supergravity modes do not have any significance in this region anyway From this perspective, it appears that the singularity at $x^+ = 0$ is an artifact of a wrong choice of dynamical variables.

5. Outlook

Raychaudhuri showed that a congruence of geodesics would tend to shrink and form singularities under generic conditions These singularities clearly signal the breakdown of General Relativity It has been suspected that a proper theory of quantum gravity would "resolve" such singularities, though it has never been clear what such a resolution might mean for space-like or null singualrities

The results from String Theory are beginning to provide a concrete meaning to this by demonstrating that space-time is itself an approximate notion which emerges from more fundamental structures which form *holographic* descriptions of gravitational physics. In this article we have described how this happens in some versions of holography, *viz.* and Matrix Theory of ten dimensional strings and the AdS/CFT correspondence In either case there is a gauge theory which provides a fundamental definition of the theory and states created by properly normalized gauge invariant operators have perfectly smooth time evolution through times which appear to be singular

Our investigations have been in toy models of cosmology However one would expect that we should be able to draw some general lessons which can be applied one day to realistic cosmology as well.

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